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# Gauged vortices in a background

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**Abstract**
We discuss the statistical mechanics of a gas of gauged vortices in the canonical formalism. At critical self-coupling, and for low temperatures, it has been argued that the configuration space for vortex dynamics in each topological class of the Abelian Higgs model approximately truncates to a finite-dimensional moduli space with a Kähler structure. For the case where the vortices live on a 2-sphere, we explain how localization formulae on the moduli spaces can be used to compute exactly the partition function of the vortex gas interacting with a background potential. The coefficients of this analytic function provide geometrical data about the Kähler structures, the simplest of which being their symplectic volume (computed previously by Manton using an alternative argument). We use the partition function to deduce simple results on the thermodynamics of the vortex system; in particular, the average height on the sphere is computed and provides an interesting effective picture of the ground state.

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## 1. Introduction

One of the most challenging aspects in the study of topological solitons in gauge field theories is to understand their interactions, even at the classical level. At critical self-coupling, where the solitons exert no net static forces among themselves, one can typically describe the dynamical interactions at low speed in terms of geodesic flow for certain metrics on the moduli spaces of stable field configurations [1]. Exact results about these metrics can be obtained in some instances, and they have provided detailed information about the classical dynamics of solitons in this regime [2]. There is now considerable evidence on the beautiful geometrical fact that these moduli spaces encode a whole range of physical information about the underlying field theories, which goes well beyond the problem of approximating the slow dynamics that brought them first into mathematical physics.

An example illustrating how physical information can be extracted from the geometry of the moduli spaces is provided by the study of the statistical mechanics of a gas of vortices in the Abelian Higgs model. Manton obtained the partition function in the critically coupled (noninteracting) regime from the volumes of the moduli spaces [3]. These volumes can be calculated once the Kähler classes of the moduli spaces are known. At first, this calculation was performed for vortices on a sphere, but a similar argument can be used to compute the partition function for vortices living on any compact Riemann surface [4].

In this paper, the Abelian Higgs model is modified by adding to the Lagrangian (at critical coupling) an external potential that will probe the interactions of the vortices themselves. For weak potentials, we expect the effect of this coupling to be well described by the addition of a potential to the moduli space dynamics. In this setting, we calculate exactly the partition function for the vortex gas in a background field. Clearly, the problem that we consider is still less ambitious than the more physically interesting (but also more difficult) statistical mechanics of vortices with Ginzburg–Landau self-interaction, but it does provide a nontrivial extension of the study by Manton. The vortices will be allowed to live on a sphere with a particular axis singled out, and the potential we shall focus on is natural given this geometry. To obtain the statistical mechanics of the system, we will be making use of a localization formula for a circle action on the moduli space of $N$ vortices. This turns out to be an alternative route to obtain Manton's results (which we recover as we switch off the interaction), in particular his formula for the volume of the moduli space of $N$ vortices on a sphere [3].

## 2. Gauged vortices and their moduli spaces

Given a Riemann surface $\Sigma$, a gauged vortex is a pair $(\mathrm{d}_A, \Phi)$ consisting of a unitary connection on a Hermitian line bundle $\mathcal{L} \to \Sigma$ and a section of this bundle, satisfying the Bogomol'nyĭ (or vortex) equations

$$\bar{\partial}_A \Phi = 0, \tag{1}$$

$$B_A = \tfrac{1}{2} * (1 - \langle \Phi, \Phi \rangle). \tag{2}$$

Here, $\langle \cdot, \cdot \rangle$ is the Hermitian structure on $\mathcal{L}$, and we have fixed a metric $\mathrm{d}s^2_\Sigma$ on $\Sigma$ with Hodge star $*$; $\bar{\partial}_A$ is defined from $\mathrm{d}_A = d - \mathrm{i}A =: \partial_A + \bar{\partial}_A$ as usual through the decomposition provided by the complex structure of $\Sigma$, and $B_A = \mathrm{d}A$ is the curvature of $\mathrm{d}_A$. These equations are invariant under gauge transformations

$$(\mathrm{d}_A, \Phi) \mapsto (\mathrm{d}_{A+\mathrm{d}\Lambda}, \mathrm{e}^{i\Lambda}\Phi), \qquad \Lambda \in C^\infty(\Sigma; \mathbb{R}) \cong \mathfrak{aut}(\mathcal{L}, \langle \cdot, \cdot \rangle).$$

The choice of Hermitian structure does not play an important role in our discussion, so once we fix local trivializations for $\mathcal{L} \to \Sigma$ we use them to pull back the standard Hermitian structure of $\mathbb{C}$.

To a configuration $(\mathrm{d}_A, \Phi)$, we associate the vortex number

$$N = \frac{1}{2\pi} \int_\Sigma B_A.$$

For the cases of interest, where $\Sigma$ is compact or is effectively compactified by imposing suitable boundary conditions, $N \in \mathbb{Z}$ and it corresponds to the degree of $\mathcal{L}$, a topological invariant. Since (1) states that $\Phi$ should be a holomorphic section, $N$ is also the number of zeros of $\Phi$, all having positive multiplicity. In fact, solutions of (1) and (2) are completely characterized by the zeros of $\Phi$, which can be any set of $N$ points on $\Sigma$ (counted with multiplicity) (cf [5]), provided that

$$4\pi N < \mathrm{Vol}(\Sigma) \tag{3}$$

holds [6, 7]. So the space of solutions to the vortex equations modulo gauge equivalence has the structure

$$\coprod_{N\in\mathbb{N}} \mathcal{M}_N,$$

where each moduli space of $N$-vortices $\mathcal{M}_N$ is the $N$th symmetric power of $\Sigma$, the smooth $2N$-manifold

$$\mathcal{M}_N = \mathrm{Sym}^N(\Sigma) := \Sigma^N/\mathfrak{S}_N.$$

Complex coordinates on this (complex) manifold are usually referred to as moduli. If $z$ is a local coordinate on an open set $U \subset \Sigma$, then the natural coordinates $(z_1, \ldots, z_N)$ on the Cartesian product $U^N$, denoting configurations with zeros of $\Phi$ at each $z = z_r$, form an appropriate set of moduli on $\mathrm{Sym}^N(U) - \Delta$, where $\Delta \subset \mathcal{M}_N$ is the locus on which at least two zeros become coincident (and thus the $\mathfrak{S}_N$-action fails to be free). The $z_r$ are interpreted as positions of $N$ individual vortex cores.

The Bogomol'nyĭ equations above appeared for the first time in the study of the Abelian Higgs model, a field theory for vortex dynamics defined by the functional

$$\mathcal{AH}_{\lambda^2}(\mathrm{D}_A, \Phi) = -\frac{1}{4}\|F_A\|_{L^2}^2 + \frac{1}{2}\|\mathrm{D}_A\Phi\|_{L^2}^2 - \frac{\lambda^2}{8}\|\langle\Phi, \Phi\rangle - 1\|_{L^2}^2 \tag{4}$$

depending on a self-coupling $\lambda^2 \in \mathbb{R}^+$. The fields $A$ and $\Phi$ here depend on a time parameter $t \in \mathbb{R}$, and the $L^2$ norms are taken with respect to the metric $\mathrm{d}t^2 - \mathrm{d}s_\Sigma^2$ on $\mathbb{R} \times \Sigma$ and the Hermitian structure on $\mathrm{pr}_\Sigma^*\mathcal{L}$; note $\mathrm{D}_A = \left(\frac{\partial}{\partial t} - \mathrm{i}A_t\right)\mathrm{d}t + \mathrm{d}_{A(t)}$ here has a time component, and we denote its curvature by $F_A = \mathrm{d}(A_t\,\mathrm{d}t + A) =: E_A \wedge \mathrm{d}t + B_A$. It was observed [8] that static configurations at critical self-coupling $\lambda^2 = 1$ are minima of the energy defined by $\mathcal{AH}_1$ if and only if they satisfy the first-order equations (1) and (2). The (potential) energy of a gauged vortex with vortex number $N$ is then $\pi N$. Since these equations are easier to study than the second-order Euler–Lagrange equations of (4), one might hope to understand the dynamics of the Abelian Higgs model from a study of gauged vortices, at least in the setting where the velocities are small, so that the field configurations are well approximated at any instant by solutions of the Bogomol'nyĭ equations [1]. One way to describe this so-called adiabatic approximation is as follows: construct from (4) an action on each $T\mathcal{M}_N$ by taking the fields to be solutions $(\mathrm{d}_{A(z)}, \Phi(z))$ of (1) and (2) with time-dependent moduli $(z_1(t), \ldots, z_N(t))$, and integrate over $\Sigma$ in the $L^2$ norms. It has been proven that the resulting mechanical system does give a good description of the true (infinite-dimensional) vortex dynamics, even when we shift slightly from critical self-coulpling [9]. Upon this process of adiabatic reduction, a potential term in the field theory becomes a potential function for the dynamics on $\mathcal{M}_N$.

The approximated Abelian Higgs dynamics on the moduli space was discussed by Samols [10] following work by Strachan [11]. It consists of geodesic motion with inertial mass $\pi$ (the static energy of one vortex) for a metric $g_{r\bar{s}}$ on $\mathcal{M}_N$. This metric is Kähler with respect to the complex structure on $\mathcal{M}_N$ induced by the one on $\Sigma$. We call it the $L^2$ metric on $\mathcal{M}_N$, since it is obtained from the natural $L^2$ norms (4) on the space of (covariant) derivatives of pairs $(\mathrm{D}_A, \Phi)$. It is described by the closed (1, 1)-form

$$\omega = \frac{\mathrm{i}}{2}\sum_{r,s=1}^{N} g_{r\bar{s}}\,\mathrm{d}z_r \wedge \mathrm{d}\bar{z}_s = \frac{\mathrm{i}}{2}\sum_{r,s=1}^{N}\left(\Omega^2(z_r)\delta_{rs} + 2\frac{\partial\bar{b}_s}{\partial z_r}\right)\mathrm{d}z_r \wedge \mathrm{d}\bar{z}_s. \tag{5}$$

Here, $z_r$ are the moduli associated with a complex coordinate $z$ for which $\mathrm{d}s^2_\Sigma = \Omega^2(z)|\mathrm{d}z|^2$. The quantities $b_r(z_1, \ldots, z_N)$ are defined as follows. One can combine (1) and (2) into a single equation for the gauge-invariant quantity $h := \log\langle\Phi, \Phi\rangle$,

$$4\frac{\partial^2 h}{\partial z \partial \bar{z}} + \Omega^2(z)(1 - \mathrm{e}^h) = 4\pi \sum_{r=1}^{N} \delta(z - z_r), \tag{6}$$

where $\delta$ is the Dirac delta-function. A solution $h(z; z_1, \ldots, z_N)$ to (6) has an expansion

$$h(z) = \log|z - z_r|^2 + a_r + \frac{1}{2}b_r(z - z_r) + +\frac{1}{2}\bar{b}_r(\bar{z} - \bar{z}_r) + \frac{\Omega^2(z_r)}{4}|z - z_r|^2 + \cdots \tag{7}$$

about a simple zero $z_r$ of $\Phi$. It is a remarkable fact that only the linear coefficients $b_r$ in this expansion appear in (5). It is clear that (7) only makes sense in a neighbourhood of the vortex $z_r$ which does not contain any other vortex positions. In fact, (6) implies that we can write [10] for each $r = 1, \ldots, N$

$$b_r(z_1, \ldots, z_N) = \sum_{\substack{s=1\\ s\neq r}}^{N} \frac{2}{z_r - z_s} + \tilde{b}_r(z_1, \ldots, z_N), \tag{8}$$

where $\tilde{b}_r$ are smooth on the coincidence locus $\Delta$. In an expansion about the position of a vortex with multiplicity $n \leqslant N$, the logarithmic term in (7) has a prefactor $n$. The local moduli $z_r$ cannot be used to describe a neighbourhood in $\mathcal{M}_N$ of such a vortex configuration, and so the coefficients in an expansion equivalent to (7) cannot be expressed as functions of the $z_r$.

Samols's formula (5) still does not give the Kähler form explicitly, since the nontrivial quantities $b_r$ are specified in terms of unknown solutions to (6). Extracting concrete information about the moduli space metrics is still a nontrivial challenge. One may feel that work in this direction must inevitably rely on a numerical study of equation (6); however, analytical results have been derived in particular cases using approximations of some sort [12, 13], integrability [11] or even a remarkable argument involving T-duality in string theory [14]. In the following, we shall obtain further analytical information about the moduli space metrics when $\Sigma$ is a 2-sphere.

## 3. Vortex dynamics in a background potential

In this paper, we would like to extend the Abelian Higgs model to include a coupling with a background potential, which we define to be any smooth function $f : \Sigma \to \mathbb{R}$. To introduce the coupling term, we first define a vorticity 2-form $v(\mathrm{D}_A, \Phi)$ on $\Sigma$ by the equation (cf [15])

$$(\mathrm{d}j + F_A) \wedge \mathrm{d}t =: v \wedge \mathrm{d}t,$$

where $j$ is the gauge-invariant supercurrent 1-form on $\mathbb{R} \times \Sigma$

$$j := \mathrm{Im}\langle\phi, \mathrm{D}_A\phi\rangle.$$

The coupling to the potential $f$ we propose is given by adding to the potential energy term of the action functional (4) the interaction term

$$\mu^2 \int_\Sigma f v \tag{9}$$

where $\mu^2$ is a coupling constant.

The Euler–Lagrange equations for the modified action

$$\mathcal{AH}_{\lambda^2}(\mathrm{D}_A, \Phi) - \mu^2 \int f v(\mathrm{D}_A, \Phi) \wedge \mathrm{d}t \tag{10}$$

are as follows. We obtain the same Gauß's law as for the Abelian Higgs model (4),

$$\mathrm{d} * E_A = 2 * \mathrm{Im}\langle \Phi, D_t \Phi \, \mathrm{d}t \rangle \tag{11}$$

(where $D_t := \frac{\partial}{\partial t} - \mathrm{i}A_t$), which is to be regarded as a constraint on the space of fields determining $A_t$. The dynamical equations are

$$\mathrm{d} * B_A - * \frac{\partial E_A}{\partial t} = - * \mathrm{Im}\langle \Phi, \mathrm{d}_A \Phi \rangle - \mu^2 (\langle \Phi, \Phi \rangle + 1) \, \mathrm{d}f, \tag{12}$$

$$\Box \Phi = \frac{\lambda^2}{2} (\langle \Phi, \Phi \rangle - 1) \Phi + 2\mathrm{i}\mu^2 * (\mathrm{d}f \wedge \mathrm{d}_A \Phi), \tag{13}$$

where $\Box := (D_t)^2 - * \mathrm{d}_A * \mathrm{d}_A$ is the covariant d'Alembertian on $\mathrm{pr}_\Sigma^* \mathcal{L} \to \mathbb{R} \times \Sigma$. As expected, there are new terms (proportional to $\mu^2$) adding to the current in Ampère's law (12), and also to the potential in the nonlinear Klein–Gordon equation (13).

From our discussion in section 2, we know that solutions to the vortex equations (1) and (2) solve the equations of motion (11) and (13) above in the static case (and setting $A_t = 0$), provided $\lambda^2 = 1$ and $\mu^2 = 0$. When the couplings are perturbed slightly away from these critical values, we expect a slow-moving solution to have a best vortex approximation, and that we can follow its evolution under the dynamics defined above. A detailed analysis of the case $\lambda^2 \simeq 1$, $\mu^2 = 0$, has been carried out by Stuart in [9], where distances to best vortex approximations were estimated (in terms of $\lambda^2 - 1$ and initial errors) after finite-time evolution, and shown to be controlled for evolution times of order $O(|\lambda^2 - 1|^{-1/2})$. In this paper, we make the natural assumption that an analogous result holds for $\lambda^2 = 1$ and $\mu^2 \simeq 0$. This assumption can be regarded as a motivation to the study of the dynamics on the moduli space of gauged vortices with a certain (and rather natural) background potential.

The coupling (9) satisfies a number of desirable properties. One of them is that, if we take $f$ to be a constant, it reduces to a constant potential on the moduli space. To see this, we use (1) and (2) to rewrite [15]

$$\begin{aligned} v &= -\mathrm{i}\langle \mathrm{d}_A \Phi, d_A \Phi \rangle + (1 - \langle \Phi, \Phi \rangle) B_A \\ &= -\mathrm{i}\langle \partial_A \Phi, \partial_A \Phi \rangle + 2(* B_A) B_A, \end{aligned}$$

which is precisely twice the energy density for solutions of the Bogomol'nyĭ equations at critical self-coupling. Thus,

$$\int_\Sigma v|_{\mathcal{M}_N} = 2\pi N.$$

Note that $\mathrm{d}f = 0$ in this situation, so the terms proportional to $\mu^2$ in the field equations (12) and (13) vanish. This is a rather degenerate case of our model (10), but the fact that the adiabatic approximation leads to sensible results at this level is already reassuring.

For the rest of the paper, we shall restrict ourselves to vortices living on a 2-sphere of radius $R$, $\Sigma = S_R^2$. In this context, we shall illustrate another natural property of the interaction term (9) in section 4.1.

## 4. Localization and the partition function

The exact results we want to derive refer to the case $\Sigma = S_R^2$, on which $z$ shall denote a stereographic coordinate. The metric on $S_R^2$ has Kähler (volume) form

$$\omega_{S_R^2} = \frac{2\mathrm{i}R^2}{(1 + |z|^2)^2} \, \mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{14}$$

and the constraint (3) reads

$$R^2 > N.$$

The moduli spaces in this case are

$$\mathcal{M}_N = \mathrm{Sym}^N\left(S^2_R\right) \cong \mathbb{CP}^N, \tag{15}$$

equipped with the Kähler structures

$$\omega = \mathrm{i}\sum_{r,s=1}^{N}\left(\frac{R^2\delta_{rs}}{(1+|z_r|^2)^2} + \frac{\partial \bar{b}_s}{\partial z_r}\right)\mathrm{d}z_r \wedge \mathrm{d}\bar{z}_s. \tag{16}$$

One way to understand (15) is as follows. Let $V_j \subset \mathcal{M}_N$ be the locus where precisely $j$ vortices are at $z = \infty$. Then $V_j$ is parametrized by the (unordered) positions of the remaining $N - j$ vortices on $\mathbb{C}$, which are unambiguously specified by the coefficients of a monic polynomial of degree $N - j$ having these positions as roots. Hence, $V_j \cong \mathbb{C}^{N-j}$. The way in which these $2(N - j)$ cells are glued together in $\mathcal{M}_N$ is determined by attaching maps corresponding to letting one vortex go to $z = \infty$ at a time. This yields precisely the description of $\mathbb{CP}^N$ as a CW-complex.

### *4.1. Rotational symmetry*

For most of our discussion, we shall restrict our attention to the potential

$$f = R^2\frac{1-|z|^2}{1+|z|^2}. \tag{17}$$

This potential is very natural if we assume that there is a special axis on the sphere. In fact, (17) is the simplest nontrivial circularly symmetric function on $S^2_R$, in the sense that other potentials with circular symmetry can be expanded as power series in $f$.

It is easy to check that (17) is a Hamiltonian for the circle action of rotations around the axis of $S^2_R$ associated with the stereographic coordinate $z$,

$$\iota_{\mathrm{i}(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}})}\omega_{S^2_R} = -\mathrm{d}f.$$

There is an induced circle action on $\mathcal{M}_N = \mathrm{Sym}^N\left(S^2_R\right)$ with generator

$$\xi = \mathrm{i}\sum_{r=1}^{N}\left(z_r\frac{\partial}{\partial z_r} - \bar{z}_r\frac{\partial}{\partial \bar{z}_r}\right) \tag{18}$$

which extends to a smooth vector field on $\mathcal{M}_N$. Rotational symmetry on $S^2_R$ implies [16]

$$\sum_{r=1}^{N}(z_r b_r - \bar{z}_r\bar{b}_r) = 0, \tag{19}$$

and one can use this equality to show that the Kähler structure (16) is preserved by the one-parameter group generated by (18). Since $\omega$ is closed, it follows from

$$0 = \mathscr{L}_\xi\omega = \iota_\xi(\mathrm{d}\omega) + \mathrm{d}(\iota_\xi\omega) = \mathrm{d}(\iota_\xi\omega)$$

and $H^1(\mathbb{CP}^N) = 0$ that there is also a Hamiltonian for the circle action on $\mathcal{M}_N$, which can be computed to be

$$J = 2\pi\sum_{r=1}^{N}\left(R^2\frac{1-|z_r|^2}{1+|z_r|^2} - (z_r b_r + 1)\right). \tag{20}$$

Note that this formula assumes that all the vortices are separated, but it does extend to a smooth function on the whole of $\mathcal{M}_N$.

Hamiltonians of circle actions are defined up to a constant. However, this constant is fixed if the circle involved is a one-parameter subgroup of a Lie group with discrete centre and Hamiltonian action on the symplectic manifold, and we demand that the Hamiltonian should be a component of the corresponding moment map. In our case, the circle action extends to a Hamiltonian action of $\mathrm{Iso}(S^2_R) = \mathrm{SO}(3)$ on $\mathcal{M}_N$. The choice of constants in (20) can be checked [16] to be consistent with the moment map $\mathcal{M}_N \to \mathfrak{so}(3)^*$. The same can be said of $f$ in (17) in relation to the symplectic structure $\omega_{S^2_R}$ in (14).

A natural question to ask at this point is whether our coupling behaves well with respect to moment maps. We can regard the adiabatic reduction of couplings of the form (9) as defining a linear map between Lie algebras

$$\begin{array}{rccc} \mathcal{R}: & C^\infty(S^2_R, \omega_{S^2_R}) & \longrightarrow & C^\infty(\mathcal{M}_N, \omega) \\ & f & \longmapsto & \int_{S^2_R} f v \end{array} \tag{21}$$

for each $N$ (where the Lie bracket on each side is the Poisson bracket defined by the symplectic structure). One may hope that this map preserves the Lie algebra structures, and that it relates corresponding components of the SO(3)-moment maps on each of $S^2_R$ and $\mathcal{M}_N$. The following proposition shows that this is indeed the case.

**Proposition 4.1.** *For each $N < R^2$, there is a commutative diagram of Lie algebras*

$$\begin{array}{ccc} & & C^\infty(S^2_R) \\ & \nearrow & \\ \mathfrak{so}(3) & & \downarrow \mathcal{R} \\ & \searrow & \\ & & C^\infty(\mathcal{M}_N) \end{array}$$

*where the diagonal arrows denote the dual maps to the moment maps $S^2_R \to \mathfrak{so}(3)^*$ and $\mathcal{M}_N \to \mathfrak{so}(3)^*$, respectively.*

**Proof.** The existence of the moment maps is guaranteed by the vanishing of the Lie algebra cohomology group $H^2(\mathfrak{so}(3); \mathbb{R})$, a consequence of simplicity [17].

Given the linearity of $\mathcal{R}$ in (21), the proposition will follow if we check the commutativity of the diagram on the generators of $\mathfrak{so}(3)$.

We start by sketching how to obtain

$$J = \mathcal{R}(f), \tag{22}$$

where $f$ is given by (17). This calculation is paradigmatic of the process of reduction to the moduli space. Suppose that the vortices are all separated, and work first on the subset

$$C_\epsilon := \left\{ z \in \mathbb{C} : |z| < \frac{1}{\epsilon} \right\} - \bigcup_{r=1}^{N} B_\epsilon(z_r) \subset S^2_R,$$

where $\epsilon$ is taken small enough. Then (6) can be used to write on $C_\epsilon$

$$v = \frac{\mathrm{i}}{2R^2}\, \mathrm{d}\left( (1+|z|^2)^2 \frac{\partial^2 h}{\partial z \partial \bar{z}} (\bar{\partial} - \partial) h \right)$$

and

$$f v = \frac{\mathrm{i}}{2R^2}\, \mathrm{d}\left( \frac{\partial}{\partial z}(1-|z|^4) \frac{\partial}{\partial z} \left( \frac{\partial h}{\partial \bar{z}} \right)^2 + \frac{\partial}{\partial z}(1+|z|^2)^2 \left( \frac{\partial h}{\partial \bar{z}} \right)^2 \right) \wedge \mathrm{d}\bar{z}.$$

Using Stokes' theorem and the estimates

$$\frac{\partial^2 h}{\partial z \partial \bar{z}} = -\frac{R^2}{(1+|z_r|^2)^2} + o(\epsilon), \qquad \frac{\partial h}{\partial \bar{z}} = \frac{1}{\epsilon}\,\mathrm{e}^{\mathrm{i}\arg(z-z_r)} + \frac{\bar{b}_r}{2} + o(1) \qquad \text{as} \quad \epsilon \to 0$$

for $z \in \partial B_\epsilon(z_r)$, which follow from (7), we do obtain (22) after taking $\epsilon \to 0$.

To proceed, we can either repeat the calculation for the other generators (as in [16]), keeping track of the constants to preserve the $\mathfrak{so}(3)$ algebra, or change integration variables and apply the appropriate rotations to the two sides of (22). The second procedure becomes straightforward once we observe that the $b_r$ transform as [4]

$$(T^* b_r)(z_1, \ldots z_N) = \frac{1}{T'(z_r)} b_r(z_1, \ldots, z_N) - \frac{T''(z_r)}{T'(z_r)^2}$$

under any holomorphic $T \in \mathrm{Iso}(\Sigma)$. This equation is readily obtained from the expansion (7). □

It should be noted that $\mathcal{R}$ does not preserve the structures of $C^\infty\left(S^2_R, \omega_{S^2_R}\right)$ and $C^\infty(\mathcal{M}_N, \omega)$ as Poisson algebras.

Since the moduli spaces are compact in our case, the circle actions we are interested in must have fixed points (the zeros of $\xi$). It is easy to see that these are exactly the $N+1$ points $p_j \in \mathcal{M}_N$ (with $j = 0, 1, \ldots, N$) describing configurations of $j$ vortices at $z = 0$ and $N - j$ vortices at $z = \infty$, which will be fixed by a rotation of all the vortices around the axis through 0 and $\infty$. We remark that $J$ is a Morse function on $\mathcal{M}_N$, with critical set

$$\mathrm{Crit}(J) = \{p_0, p_1, \ldots, p_N\}.$$

In the next section, it will be useful to understand the circle action in the neighbourhood of the fixed points. On each tangent space $T_{p_j}\mathcal{M}_N \cong \mathbb{C}^N$, the action linearizes to a complex $N$-dimensional representation of the circle group; this in turn decomposes into $N$ one-dimensional representations of U(1), and each of them is uniquely specified by its weight $k \in \mathbb{Z}$:

$$\mathrm{e}^{2\pi \mathrm{i} t} : \zeta \mapsto \mathrm{e}^{2\pi \mathrm{i} k t}\zeta, \qquad t \in \mathbb{R}, \quad \zeta \in \mathbb{C}.$$

Thus to each of the fixed points $p_j$ we can associate $N$ weights $k_{j,\ell}$, $1 \leqslant \ell \leqslant N$, well defined up to order. The product of the weights at each fixed point

$$e(p_j) := \prod_{\ell=1}^{N} k_{j,\ell} \tag{23}$$

is a local invariant of the circle action: it does not depend on the choice of coordinates on the moduli space. We shall make use of the following result:

**Lemma 4.2.** *For the circle action generated by (20) on* $(\mathcal{M}_N, \omega)$,

$$e(p_j) = (-1)^{j+N} j!(N-j)!, \qquad 0 \leqslant j \leqslant N.$$

**Proof.** In terms of the moduli $z_r$, the circle action is simply given by

$$\mathrm{e}^{2\pi \mathrm{i} t} : z_r \mapsto \mathrm{e}^{2\pi \mathrm{i} t} z_r, \qquad t \in \mathbb{R}. \tag{24}$$

However, all the $p_j$ except $p_0$ lie in $\Delta$, where the coordinate system defined by the $z_r$ becomes singular, and so we must introduce other coordinates to compute the weights. We fix $j$ and arrange the vortex labels such that vortices $1, \ldots, j$ are at $z = 0$, and vortices $j+1, \ldots, N$ are at $z = \infty$. (We are allowed to do this since the vortices in each cluster are to be thought of

as interchangeable, but vortices belonging to different clusters have separate identities.) Now we introduce

$$u_{j,r} := s_r^{[j]}(z_1, \ldots, z_j), \qquad 1 \leqslant r \leqslant j,$$
$$v_{j,r} := s_r^{[N-j]}\big(z_{j+1}^{-1}, \ldots, z_N^{-1}\big), \qquad 1 \leqslant r \leqslant N-j,$$

where $s_r^{[j]}$ denotes the $r$th elementary symmetric polynomial in $j$ variables,

$$s_r^{[j]}(t_1, \ldots, t_j) := \sum_{i_1 < \cdots < i_r} t_{i_1} \cdots t_{i_r}.$$

Clearly, $\{u_{j,1}, \ldots, u_{j,j}, v_{j,1}, \ldots, v_{j,N-j}\}$ is a centred local coordinate system at $p_j \in \mathcal{M}_N$, and we can also use it as a coordinate system on $T_{p_j}\mathcal{M}_N$. From (24), we find that the linearization of the circle action is described by

$$u_{j,r} \mapsto \mathrm{e}^{2\pi \mathrm{i} r t} u_{j,r} \qquad v_{j,r} \mapsto \mathrm{e}^{2\pi \mathrm{i}(-r)t} v_{j,r}$$

and therefore we obtain in (4.2)

$$e(p_j) = \left(\prod_{k=1}^{j} k\right)\left(\prod_{\ell=1}^{N-j}(-\ell)\right) = (-1)^{j+N} j!(N-j)!.$$

□

*4.2. The partition function*

The dynamics on the moduli space of $N$ vortices is defined by the Lagrangian

$$L = \frac{\pi}{2}\sum_{r,s=1}^{N} g_{r\bar{s}}(z_1, \ldots, z_N)\dot{z}_r\dot{\bar{z}}_s - \mu^2 J(z_1, \ldots, z_N),$$

where $g_{r\bar{s}}$ are the coefficients of the metric on $\mathcal{M}_N$ and $\pi$ is the mass of a single vortex. In the canonical picture, we describe the dynamics as a Hamiltonian system on the $4N$-dimensional manifold $T^*\mathcal{M}_N$ equipped with its canonical symplectic structure

$$\omega_{\mathrm{can}} = \frac{1}{2}\sum_{r=1}^{N}(\mathrm{d}z_r \wedge d\bar{w}_r + \mathrm{d}\bar{z}_r \wedge dw_r).$$

Here, $w_r$ are complex coordinates for the fibres of $T^*\mathcal{M}_N \to \mathcal{M}_N$, conjugate to the moduli $z_r$:

$$w_r = \frac{\partial L}{\partial \dot{z}_r} = \pi \sum_{s=1}^{N} g_{r\bar{s}}\dot{\bar{z}}_s.$$

The dynamics is generated by the Hamiltonian

$$H = \frac{1}{2\pi}\sum_{r,s=1}^{N} g^{r\bar{s}}(z_1, \ldots, z_N) w_r \bar{w}_s + \mu^2 J(z_1, \ldots, z_N),$$

where $g^{r\bar{s}}$ denote the entries of the inverse to the matrix of coefficients of the metric.

According to the canonical formalism for classical statistical mechanics, the partition function of the vortex moduli space dynamics is given by the Gibbs formula

$$Z = \frac{1}{(2\pi\hbar)^{2N}}\int_{T^*\mathcal{M}_N} \exp(-H(z_1, \ldots, z_N, w_1, \ldots, w_N)/T)\frac{\omega_{\mathrm{can}}^{2N}}{(2N)!}. \tag{25}$$

The prefactor to the integral is Planck's constant $2\pi\hbar$ raised to the power $\frac{1}{2}\dim_{\mathbb{R}} T^*\mathcal{M}_N = 2N$, and $T$ is the temperature. We are normalizing Boltzmann's constant to unity.

In parallel with the calculation in [3], we find that the integral (25) factorizes as a product of a Gaussian integral along the fibres, which can be readily calculated, and an integral over the moduli space with the Liouville measure associated with (16)

$$Z = \left(\frac{T}{2\hbar^2}\right)^N \int_{\mathcal{M}_N} \exp(-\mu^2 J(z_1, \ldots, z_N)/T) \frac{\omega^N}{N!}. \tag{26}$$

The integral remaining still looks very complicated, but we shall show that it can also be computed exactly, using localization in symplectic geometry. The main tool we will use is the following version of a famous result by Duistermaat and Heckman [18]:

**Theorem 4.3** (Duistermaat–Heckman formula)**.** *Let $(M, \omega)$ be a $(2n)$-dimensional compact symplectic manifold with a Hamiltonian circle action generated by a Morse function $K : M \to \mathbb{R}$. Then,*

$$\int_M \mathrm{e}^{\tau K} \frac{\omega^n}{n!} = \sum_{p \in \mathrm{Crit}(K)} \frac{\mathrm{e}^{\tau K(p)}}{\tau^n e(p)}, \tag{27}$$

*where $e(p)$ denotes the product of the weights of the linearized action at the critical point $p$ and $\tau$ is a formal parameter.*

A streamlined proof of this theorem can be found in section 5.6 of reference [19].

In our context, taking $M = \mathcal{M}_N, K = J$ and $\tau = -\mu^2/T$, and making use of lemma 4.2, the equality (27) takes the form

$$\int_{\mathcal{M}_N} \mathrm{e}^{-\mu^2 J/T} \frac{\omega^N}{N!} = \sum_{j=0}^{N} (-1)^j \frac{T^N \exp(-\mu^2 J(p_j)/T)}{\mu^{2N} j!(N-j)!}. \tag{28}$$

Hence, to compute the integral in (26), we only need to evaluate the potential $J$ at the critical points $p_j \in \mathcal{M}_N$.

We shall make use of the spherical symmetry of the problem to determine the contribution of the $\tilde{b}_r$ terms in formula (20) to $J(p_j)$. Suppose that the Higgs field $\Phi$ has a zero at $z = y$ of order $j$ and a zero at $z = -\frac{1}{\bar{y}}$ of order $N - j$, a vortex configuration corresponding to a point of $\mathcal{M}_N$ that we shall denote by $p_j(y)$. Then $h = \log\langle\Phi, \Phi\rangle$ must satisfy

$$\frac{\partial^2 h}{\partial z \partial \bar{z}} - \frac{R^2(\mathrm{e}^h - 1)}{(1+|z|^2)^2} = j\pi\delta(z-y) + (N-j)\pi\delta\left(z - \frac{1}{\bar{y}}\right). \tag{29}$$

This equation leads to the following expansions for $h$:

$$h(z, y) = j \log|z-y|^2 + a_+(y) + \tfrac{1}{2} b_+(y)(z-y) + \tfrac{1}{2}\overline{b_+(y)}(\bar{z} - \bar{y}) + \cdots$$

around $z = y$, and

$$h(z, y) = (N-j)\log\left|z + \frac{1}{\bar{y}}\right|^2 + a_-(y) + \frac{1}{2} b_-(y)\left(z + \frac{1}{\bar{y}}\right) + \frac{1}{2}\overline{b_-(y)}\left(\bar{z} + \frac{1}{y}\right) + \cdots$$

around $z = -\frac{1}{\bar{y}}$. We need to calculate $b_\pm(y)$. Extending an argument in [3], we explore the fact that $h(z, y)$ must be a function of the spherical (and thus also of the chordal) distance between the points of coordinates $z$ and $y$ on $S^2_R$. The square of the chordal distance is given by

$$\frac{4R^2|z-y|^2}{(1+|z|^2)(1+|y|^2)},$$

and so $h$ can be expanded as

$$h(z, y) = j \log \frac{|z-y|^2}{(1+|z|^2)(1+|y|^2)} + c_+ + d_+ \frac{|z-y|^2}{(1+|z|^2)(1+|y|^2)} + \cdots. \tag{30}$$

For $y = 0$, this yields

$$h(z, 0) = j \log|z|^2 + c_+ + (d_+ - j)|z|^2 + \cdots;$$

substituting this into (29) with $y = 0$ and small $z$, and looking at the zeroth-order term in $|z|^2$, we conclude that

$$d_+ = j - R^2.$$

Now taking $y$ arbitrary, we rearrange (30) as

$$\begin{aligned} h(z, y) = & j \log|z - y|^2 + c_+ - 2j \log(1 + |y|^2) - \frac{j\bar{y}}{1 + |y|^2}(z - y) - \frac{jy}{1 + |y|^2}(\bar{z} - \bar{y}) \\ & + \frac{j\bar{y}^2}{(1 + |y|^2)^2}(z - y)^2 + \frac{jy^2}{(1 + |y|^2)^2}(\bar{z} - \bar{y})^2 - \frac{R^2}{(1 + |y|^2)^2}|z - y|^2 + \cdots \end{aligned}$$

to read off

$$b_+(y) = -\frac{2j\bar{y}}{1 + |y|^2}. \tag{31}$$

We proceed similarly for $b_-(y)$: writing

$$h(z, y) = (N - j) \log \frac{\left|z + \frac{1}{\bar{y}}\right|^2}{(1 + |z|^2)\left(1 + \frac{1}{|y|^2}\right)^2} + c_- + d_- \frac{\left|z + \frac{1}{\bar{y}}\right|^2}{(1 + |z|^2)\left(1 + \frac{1}{|y|^2}\right)^2} + \cdots \tag{32}$$

we can calculate

$$c_- = N - j - R^2$$

by taking $y = \infty$, substituting into (29) and looking at the zeroth term in the large $|z|^2$ expansion; then rearrange (32) as

$$\begin{aligned} h(z, y) = & (N - j) \log \left|z + \frac{1}{\bar{y}}\right|^2 - 2(N - j) \log \frac{1 + |y|^2}{|y|} + c_- + \frac{(N - j)\bar{y}}{1 + |y|^2}\left(z + \frac{1}{\bar{y}}\right) \\ & + \frac{(N - j)y}{1 + |y|^2}\left(\bar{z} + \frac{1}{y}\right) + \frac{(N - j)\bar{y}^2}{(1 + |y|^2)^2}\left(z + \frac{1}{\bar{y}}\right)^2 \\ & + \frac{(N - j)y^2}{(1 + |y|^2)^2}\left(\bar{z} + \frac{1}{y}\right)^2 - \frac{R^2|y|^2}{(1 + |y|^2)^2}\left|z + \frac{1}{\bar{y}}\right|^2 + \cdots \end{aligned}$$

and read off

$$b_-(y) = \frac{2(N - j)\bar{y}}{1 + |y|^2}. \tag{33}$$

To proceed, we must interpret formula (20) carefully to deal with vortex clusters such as the configurations $p_j(y) \in \mathcal{M}_N$. When any number of vortices become coincident, the smooth part $\tilde{b}_r$ of their individual $b_r$ coefficients tends to the linear coefficient in the cluster expansion of $h$ about the position of the cluster, but the singularities in (8) must be treated with some care. Although these singular parts diverge separately, they always yield a finite contribution in a $\mathfrak{S}_N$-invariant linear combination over the individual vortices—for example, they cancel mutually in a term like $\sum_{r=1}^{N} b_r$. In (20), these singular parts are multiplied by the vortex positions, and so in the clustering $(z_1, \ldots, z_N) \to p_j(y)$ they give a contribution

$$
\begin{aligned}
-2\pi \sum_{r=1}^{N} z_r \sum_{\substack{s\neq r\\ \text{same cluster}}} \frac{2}{z_r - z_s} &= -4\pi \sum_{r=1}^{j}\sum_{s\neq r} \frac{z_r}{z_r - z_s} - 4\pi \sum_{r=j+1}^{N}\sum_{s\neq r} \frac{z_r}{z_r - z_s} \\
&= -4\pi \left( \sum_{r=1}^{j}\sum_{s<r} + \sum_{r=j+1}^{N}\sum_{s<r} \right) \left( \frac{z_r}{z_r - z_s} + \frac{z_s}{z_s - z_r} \right) \\
&= -4\pi \left( \sum_{r=1}^{j}\sum_{s<r} + \sum_{r=j+1}^{N}\sum_{s<r} \right) 1 \\
&= -4\pi \left( \frac{j(j-1)}{2} + \frac{(N-j)(N-j-1)}{2} \right) \\
&= -2\pi((N-j)^2 + j^2 - N) \qquad (34)
\end{aligned}
$$

to $J(p_j(y))$. The remaining part of $J(p_j(y))$ is

$$
\begin{aligned}
2\pi \sum_{r=1}^{N} \left( R^2 \frac{1-|z_r|^2}{1+|z_r|^2} - (z_r \tilde{b}_r + 1) \right) &= 2\pi R^2 \left( j \frac{1-|y|^2}{1+|y|^2} + (N-j) \frac{1-\frac{1}{|y|^2}}{1+\frac{1}{|y|^2}} \right) \\
&\quad - 2\pi \left( j(y b_+(y) + 1) + (N-j) \left( \left( -\frac{1}{\bar{y}} \right) b_-(y) + 1 \right) \right) \\
&= 2\pi(2j-N)(R^2-N) \frac{1-|y|^2}{1+|y|^2} + 2\pi((N-j)^2 + j^2 - N), \qquad (35)
\end{aligned}
$$

where we used (31) and (33). Adding (34) and (35), we obtain

$$
J(p_j(y)) = 2\pi(R^2 - N)(2j - N) \frac{1-|y|^2}{1+|y|^2}
$$

and hence

$$
J(p_j) = J(p_j(0)) = 2\pi(R^2 - N)(2j - N), \qquad 0 \leqslant j \leqslant N. \qquad (36)
$$

Inserting (36) in (28), we find

$$
\int_{\mathcal{M}_N} \mathrm{e}^{-\mu^2 J/T} \frac{\omega^N}{N!} = \sum_{j=0}^{N} (-1)^j \frac{T^N \exp(-2\pi\mu^2(R^2-N)(2j-N)/T)}{\mu^{2N} j!(N-j)!}. \qquad (37)
$$

This equation can be interpreted as an equality in the formal power series ring $\mathbb{R}\left[\left[\frac{\mu^2}{T}\right]\right]$, which is equivalent to an infinite number of identities over the reals. The first $N$ nontrivial identities are

$$
\sum_{j=0}^{N} \frac{(-1)^j}{j!(N-j)!} (2j-N)^k = 0, \qquad 0 \leqslant k \leqslant N-1, \qquad (38)
$$

and they must be true for consistency. But they are implied by the following technical lemma.

**Lemma 4.4.** *For all $N \in \mathbb{N}$,*

$$
\sum_{j=0}^{N} \frac{(-1)^j}{N!} \binom{N}{j} \left( \frac{N}{2} - j \right)^k = \begin{cases} 0 & \text{if} \quad 0 \leqslant k \leqslant N-1, \\ 1 & \text{if} \quad k = N. \end{cases}
$$

**Proof.** We start from the equality

$$
\left( x \frac{\mathrm{d}}{\mathrm{d}x} \right)^{\ell} (1-x)^N = \sum_{j=0}^{N} (-1)^j \binom{N}{j} j^{\ell} x^j, \qquad \ell \in \mathbb{N} \cup \{0\},
$$

obtained by successively acting on the binomial expansion with the Euler operator $x\frac{\mathrm{d}}{\mathrm{d}x}$. Setting $x = 1$ yields

$$\sum_{j=0}^{N}(-1)^j\binom{N}{j}j^\ell = \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell (1-x)^N\bigg|_{x=1}.$$

We use this to write

$$\begin{aligned}\sum_{j=0}^{N}(-1)^j\binom{N}{j}\left(\frac{N}{2}-j\right)^k &= \sum_{j=0}^{N}\sum_{\ell=0}^{k}(-1)^{j+\ell}\binom{N}{j}\binom{k}{\ell}\left(\frac{N}{2}\right)^{k-\ell} j^\ell \\ &= \sum_{\ell=0}^{N}(-1)^\ell\binom{k}{\ell}\left(\frac{N}{2}\right)^{k-\ell}\sum_{j=0}^{N}(-1)^j\binom{N}{j}j^\ell \\ &= \sum_{\ell=0}^{N}(-1)^\ell\binom{k}{\ell}\left(\frac{N}{2}\right)^{k-\ell}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell (1-x)^N\bigg|_{x=1} \\ &= \left(\frac{N}{2}-x\frac{\mathrm{d}}{\mathrm{d}x}\right)^k (1-x)^N\bigg|_{x=1}. \end{aligned} \tag{39}$$

It is clear that (39) is zero whenever $0 \leqslant k < N$, since the differential operator $\left(\frac{N}{2} - x\frac{\mathrm{d}}{\mathrm{d}x}\right)^k$ will not annihilate enough $(1-x)$ factors before $x \to 1$. However, if $k = N$ one obtains

$$\begin{aligned}\left(\frac{N}{2}-x\frac{\mathrm{d}}{\mathrm{d}x}\right)^N (1-x)^N\bigg|_{x=1} &= (-1)^N\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^N (1-x)^N\bigg|_{x=1} \\ &= (-1)^N x^N\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^N (1-x)^N\bigg|_{x=1} \\ &= N!\end{aligned}$$

making use of $\left[\frac{\mathrm{d}}{\mathrm{d}x}, x\right] = 1$. □

Note that lemma 4.4 yields yet another identity from (37):

$$\begin{aligned}\mathrm{Vol}(\mathcal{M}_N) := \int_{\mathcal{M}_N}\frac{\omega^N}{N!} &= \frac{(4\pi)^N(R^2-N)^N}{N!}\sum_{j=0}^{N}\frac{(-1)^j}{j!(N-j)!}\left(\frac{N}{2}-j\right)^N \\ &= \frac{(4\pi)^N(R^2-N)^N}{N!}.\end{aligned} \tag{40}$$

This is precisely the formula found by Manton for the volume of the vortex moduli space in [3], using a more direct argument involving the cohomology of $\mathbb{CP}^N$. Equation (40) provides a nontrivial check of our calculations.

Beyond the identities (38) and formula (40) for $\mathrm{Vol}(\mathcal{M}_N)$, our localization argument yields an infinite number of integrals over the moduli space: for $m \in \mathbb{N}$

$$\int_{\mathcal{M}_N} J(z_1,\ldots,z_N)^m\frac{\omega^N}{N!} = \sum_{j=0}^{N}\frac{(-1)^{N-j}m!}{j!(N-j)!(N+m)!}(2\pi(R^2-N)(2j-N))^{N+m}. \tag{41}$$

Note that both sides of this equation vanish when $m$ is odd: the left-hand side from reflection symmetry on $S^2_R$ and proposition 4.1, and the right-hand side from

$$\sum_{j=0}^{N}\frac{(-1)^j}{j!(N-j)!}\left(\frac{N}{2}-j\right)^{N+2n-1} = 0, \qquad \forall n \in \mathbb{N},$$

which follows from substituting $j$ by $N-j$ into the sum. For $m$ even, (41) yields new quantitative information about the metric on $\mathcal{M}_N$. We note in passing that even the vanishing integrals ($m$ odd) have interesting content. For example, using the result for $m=1$, we find from (20) that

$$\int_{\mathcal{M}_N} \sum_{r=1}^{N} z_r b_r(z_1,\ldots,z_N)\frac{\omega^N}{N!} = -\mathrm{Vol}(\mathcal{M}_N);$$

a similar argument for rotations around $z=1$ and $z=\mathrm{i}$ yields

$$\int_{\mathcal{M}_N} \sum_{r=1}^{N} b_r(z_1,\ldots,z_N)\frac{\omega^N}{N!} = 0,$$

which in turn also leads to

$$\int_{\mathcal{M}_N} \sum_{r=1}^{N} z_r^2 b_r(z_1,\ldots,z_N)\frac{\omega^N}{N!} = 0$$

by a property of the functions $b_r$ analogous to (19) [16]:

$$\sum_{r=1}^{N}\left(2z_r + z_r^2 b_r + \bar{b}_r\right) = 0.$$

Using the result (41), together with (40), we can obtain the integral over $\mathcal{M}_N$ of any power series in $J$. Such power series, when convergent, give analytic functions on $\mathcal{M}_N$ which are invariant under the circle action generated by (18).

Finally, we can use (37) to compute the partition function (26) of the vortex gas in the background field (17) to be

$$\begin{aligned} Z &= \left(\frac{T^2}{2\hbar^2\mu^2}\right)^N \exp(2\pi\mu^2 N(R^2-N)/T)\sum_{j=0}^{N}\frac{(-1)^j}{j!(N-j)!}\exp(-4\pi\mu^2(R^2-N)j/T) \\ &= \frac{1}{N!}\left(\frac{T^2}{2\hbar^2\mu^2}\right)^N \exp(2\pi\mu^2 N(R^2-N)/T)(1-\exp(-4\pi\mu^2(R^2-N)/T))^N \qquad (42) \\ &= \frac{1}{N!}\left(\frac{T}{\hbar\mu}\right)^{2N}\sinh^N\left(\frac{2\pi\mu^2(R^2-N)}{T}\right). \end{aligned}$$

## 5. Thermodynamics of the vortex gas

The Helmholtz free energy of the vortex system can be computed from (42) as

$$F = -T\log Z \simeq -NT\left(\log\sinh\frac{\mu^2(A-4\pi N)}{2T} - \log N + 2\log\frac{\sqrt{e}T}{\hbar\mu}\right),$$

where we made use of Stirling's approximation $\ln N! \simeq N\ln N - N$, and we introduced the area of $S_R^2$, $A=4\pi R^2$. The entropy is given by

$$S = -\frac{\partial F}{\partial T} = N\left(\log\sinh\frac{\mu^2(A-4\pi N)}{2T} + \log\frac{\mathrm{e}^3T^2}{\hbar^2\mu^2 N} - \frac{\mu^2(A-4\pi N)}{2T}\coth\frac{\mu^2(A-4\pi N)}{2T}\right).$$

Both these quantities turn out to be nonextensive, due to a nonlinear effect produced by the interaction with the external potential.

The interaction is controlled by the coupling $\mu^2$; at small coupling, keeping $A$ and $N$ finite, we can approximate the hyperbolic functions to first order as $\sinh\chi \simeq \chi$ and

$\coth\chi \simeq \frac{1}{\chi}$, leading to the same results found by Manton in the absence of interaction [3]. In this noninteracting setting, both $F$ and $S$ become extensive in the thermodynamical limit of $A \to \infty$, $N \to \infty$ and constant density $n = \frac{N}{A}$. The pressure $P = -\frac{\partial F}{\partial A}$ of the system in this regime can be readily computed and yields the equation of state [3]

$$P(A - 4\pi N) = NT. \tag{43}$$

This is a particular limit of the van der Waals equation, known as a Clausius equation of state. It holds more generally on any compact Riemann surface [4]. The fact that the factor $A - 4\pi N$ appears in (43) can be interpreted as an interaction among the vortices [3]: each vortex effectively occupies an area of $4\pi$ (consistently with (3)), hence $N$ coexisting vortices have an area available for their motion which is a reduction of the area $A$ of the sphere by $N \times 4\pi$. The virial coefficients associated with (43) are found to be all constant and equal to powers of $4\pi$:

$$PA = NT \sum_{\ell=0}^{\infty} (4\pi)^{\ell} n^{\ell}; \tag{44}$$

this virial expansion is reminiscent of the one for a gas of hard particles of finite size in a one-dimensional box [2]. Thus, one might be tempted to think of the vortices in effective terms as rigid discs of area $4\pi$ moving on the surface. But this picture already fails at first order in the expansion (44): the gas of hard discs would have first virial coefficient $8\pi$ [20], which is twice the coefficient of $n$ in the power series in (44). In fact, a crucial difference between the gas of vortices and a gas of hard discs is that the shapes of the regions where the vortex density is concentrated become very different in the two cases whenever two or more particles come close together, and we expect the equation of state to be extremely sensitive to this.

Now we shall consider the statistical mechanics of the interacting regime ($\mu^2 \neq 0$). From our partition function (42), we can compute the thermodynamic average value of the observable $J \in C^{\infty}(\mathcal{M}_N)$ given by (20):

$$\begin{aligned}\langle J \rangle &= \frac{1}{Z} \int_{T^*\mathcal{M}_N} J\, \mathrm{e}^{-H/T} \frac{\omega_{\mathrm{can}}^{2N}}{(2N)!} \\ &= -T \frac{\partial}{\partial \mu^2} \log Z \\ &= -NT \left( \frac{A - 4\pi N}{2T} \coth \frac{\mu^2 (A - 4\pi N)}{2T} - \frac{1}{\mu^2} \right).\end{aligned} \tag{45}$$

Recall that $J$ is related to the height function on the sphere $x_3 := R\frac{1-|z|^2}{1+|z|^2} \in C^{\infty}(S_R^2)$ by $J = \mathcal{R}(Rx_3)$ (cf (22)). Thus, we propose to interpret the quantity $J/(2\pi N R)$ as an observable giving the height on $S_R^2$ of configurations of $N$ vortices, and we shall denote it by $\tilde{x}_3$. We find

$$\langle \tilde{x}_3 \rangle = \frac{\langle J \rangle}{2\pi N R} = -\left(1 - \frac{N}{R^2}\right) R \left( \coth\chi - \frac{1}{\chi} \right), \tag{46}$$

where

$$\chi := \frac{\mu^2 (A - 4\pi N)}{2T}. \tag{47}$$

The dependence of the average height on the parameter $\chi$ is plotted in figure 1.

To interpret the meaning of (46), consider a simple model where the vortex density is supported and homogeneously distributed on a spherical disc of area $4\pi N$ and centred at the

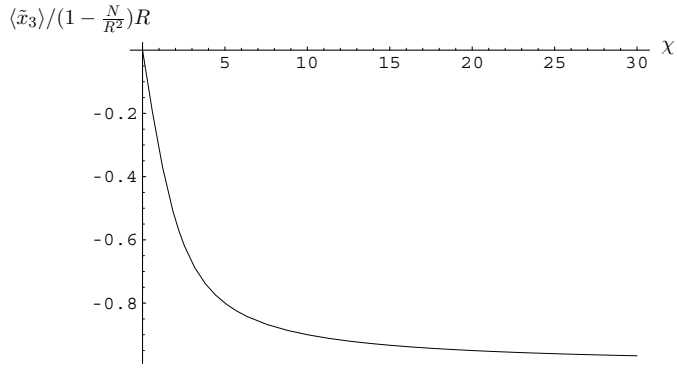


**Figure 1.** The thermodynamic average $\langle\tilde{x}_3\rangle$ of the height observable in the gas of $N$ vortices as a function of the parameter $\chi$.

minimum $z = \infty$ of the potential (17). The height $x_3^{\max}$ of the points at the boundary of this disc is computed as

$$2\pi R \int_{-R}^{x_3^{\max}} \mathrm{d}x_3 = 4\pi N \Rightarrow x_3^{\max} = -R + \frac{2N}{R},$$

where we made use of Archimedes' hat-box theorem. In this model, the average height for the vortex density is then given by

$$\langle x_3\rangle^{\text{model}} := \frac{\int_{-R}^{x_3^{\max}} x_3\,\mathrm{d}x_3}{\int_{-R}^{x_3^{\max}} \mathrm{d}x_3} = -\left(1 - \frac{N}{R^2}\right) R. \tag{48}$$

Thus, we find that

$$\lim_{\chi\to\infty} \langle\tilde{x}_3\rangle = \langle x_3\rangle^{\text{model}}.$$

In other words, the effective shape of an $N$-vortex distorts to a spherical disc (of area $4\pi N$ and centred at the minimum of the potential $z = \infty$) as $\chi$ becomes large. This limit is typically achieved if the area $A - 4\pi N$ available for the dynamics is large, the coupling $\mu^2$ is large or the absolute temperature $T$ is low—cf (47). Note that in the picture where each vortex is approximated by a rigid disc of area $4\pi$, the average height of the density distribution would be higher than (48). The picture of the $N$-vortex as a disc of constant density (localized at the bottom of the potential) provides a good description of the ground state of the system.

To describe a circularly symmetric distribution of vortices on the sphere, it is natural to introduce a vortex density function $\rho : [-R, R] \to \mathbb{R}$, defined on the interval of heights on $S_R^2$ and normalized as $\int_{-R}^{R} \rho\,\mathrm{d}x_3 = 1$. This function has parameters $A$, $T$ and $N$, and it could in principle be reconstructed as a Fourier–Legendre series from the partition function (42) if we could compute the reduction to the moduli space of all the powers of the height, $\mathcal{R}(f^m)$ for $m \in \mathbb{N}$. We can write

$$\int_{-R}^{R} x_3\rho(x_3)\,\mathrm{d}x_3 := \langle\tilde{x}_3\rangle = \frac{\langle J\rangle}{2\pi N R},$$

which gives just the first Fourier–Legendre coefficient. The difference in pressure $\Delta P$ between the highest and the lowest points of the gas on the sphere depends only on the trivial zeroth-order coefficient: using the relation

$$\nabla P + N\rho\nabla f = 0$$

for the pressure of a fluid of $N$ particles subject to a potential $f$ at equilibrium (this is analogous to the problem of fluid motion in a constant gravitational field, cf [21] § 25), we find

$$\Delta P = \int_{-R}^{R} \nabla P \, \mathrm{d}x_3 = -\int_{-R}^{R} N\rho(x_3) \frac{\partial}{\partial x_3}(\mu^2 R x_3) \, \mathrm{d}x_3 = -\mu^2 N R < 0.$$

## 6. Discussion

We have been able to calculate the partition function of a gas of critically coupled Abelian Higgs vortices interacting with an axially symmetric background potential on a sphere. When we switch off the interaction, we recover Manton's partition function that gives physical insight on the Liouville volume of vortex moduli spaces $\mathcal{M}_N$. Our study yields, in addition to Manton's formula (40) for $\mathrm{Vol}(\mathcal{M}_N)$, an infinite number of nontrivial integrals (41) over $\mathcal{M}_N$. These are additional data about the geometry of the moduli spaces, and they are all encapsulated by our partition function. As an application, we computed the thermodynamic average height of the gas of $N$ vortices, and the result we found is consistent with the effective picture of the ground state as an $N$-vortex localized at the bottom of the potential as a spherical disc of constant density and area $4\pi N$.

Our analysis was essentially an application of the Duistermaat–Heckman localization formula for a natural circle action on $(\mathcal{M}_N, \omega)$, in which the symplectic structure of the moduli space is a crucial ingredient. There is an alternative model [22] for the dynamics of Ginzburg–Landau vortices with a Schrödinger–Chern–Simons kinetic term, for which $\mathcal{M}_N$ (not $T^*\mathcal{M}_N$) plays the role of phase space in the adiabatic approximation; in this context, the Kähler form $\omega$ appears naturally as a symplectic structure [16]. Our work illustrates that the symplectic point of view can also be fruitful in the study of the Abelian Higgs model.

We have already noted that our formula (41) can be applied to calculate integrals of general circularly symmetric functions on the moduli space. One may therefore hope to study the interaction of the vortices with any symmetric potential $f$ on $S_R^2$—or perhaps even an SO(3)-invariant intervortex interaction modelling the Ginzburg–Landau potential at $\lambda^2 \neq 1$ to some degree of approximation, which would be an obviously interesting extension of our work [23]. Analytical results about the Ginzburg–Landau potential (for arbitrary $N$) are already available [24, 25], but they refer to the situation where the vortices are well separated on the plane. A treatment of the interaction to include the interesting effects near clustering configurations on the moduli spaces will almost certainly need to use some numerical input [26]. It is believed that Abelian Higgs vortices even slightly away from critical coupling should satisfy a realistic equation of state such as the van der Waals equation, which accounts for phase transitions. Progress in this direction would shed light on the phenomenology of thin superconductors.

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